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Journal of Approximation Theory 123 (2003) 162–172

JOURNAL OF Approximation Theory

http://www.elsevier.com/locate/jat

Ratio asymptotics for orthogonal rational functions on an interval☆

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Received 30 November 2001; accepted in revised form 21 May 2003

Communicated by Guillermo López Lagomasino

Abstract

Let $\{\alpha_1, \alpha_2, \ldots\}$ be a sequence of real numbers outside the interval $[-1, 1]$ and μ a positive bounded Borel measure on this interval satisfying the Erdős–Turán condition $\mu' > 0$ a.e., where μ' is the Radon–Nikodym derivative of the measure μ with respect to the Lebesgue measure. We introduce rational functions $\varphi_n(x)$ with poles $\{\alpha_1, \ldots, \alpha_n\}$ orthogonal on $[-1, 1]$ and establish some ratio asymptotics for these orthogonal rational functions, i.e. we discuss the convergence of $\varphi_{n+1}(x)/\varphi_n(x)$ as n tends to infinity under certain assumptions on the location of the poles. From this we derive asymptotic formulas for the recurrence coefficients in the three-term recurrence relation satisfied by the orthonormal functions.

Keywords: Orthogonal rational functions; Orthogonal polynomials; Ratio asymptotics

1. Introduction

Ratio asymptotics for orthogonal polynomials on the interval $[-1, 1]$ have been studied in several books and papers. Most results were obtained relating these polynomials to polynomials orthogonal on the unit circle in the complex plane using

☆This work is partially supported by the Fund for Scientific Research (FWO), project "CORFU: Constructive study of orthogonal functions", Grant #G.0184.02 and the Belgian Programme on Interuniversity Poles of Attraction, initiated by the Belgian State, Prime Minister's Office for Science, Technology and Culture. The scientific responsibility rests with the authors.

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0021-9045/03/$ - see front matter
doi:10.1016/S0021-9045(03)00116-3

the Joukowski transform $x = \frac{1}{2}(z + z^{-1})$ which maps the unit circle to the interval $[-1, 1]$. In this way, Szegő [9] obtained convergence results for weights satisfying Szegő's condition, and later Rakhmanov [6,7] derived results for the weaker Erdős–Turán condition. See also [4,5] for results about polynomials orthogonal with respect to varying measures.

Orthogonal rational functions are a generalization of orthogonal polynomials, in such a way that we recover the polynomial situation if all poles are at infinity. Asymptotics for rational functions orthogonal on the unit circle (or, using a Cayley transform, on the extended real line) were studied in [1], but the case of a finite interval has thus far not been treated. In this paper, it is our aim to derive convergence results for orthogonal rational functions on the interval $[-1, 1]$, using a relation between rational functions on the unit circle and the interval $[-1, 1]$ as described in [10].

Just as in the polynomial case, orthogonal rational functions satisfy a three-term recurrence relation. Asymptotics for the recurrence coefficients can be derived in a very natural way from the ratio asymptotics for the orthogonal functions, as in [6].

In the next sections, we introduce the spaces of rational functions we are dealing with and we discuss the recurrence relation. We will need several results before we can state our main theorem about the convergence of the ratio of orthogonal rational functions in Section 6. Using this theorem it is not difficult to derive asymptotics for the recurrence coefficients.

2. Preliminaries

The complex plane is denoted by $\mathbb{C}$, the Riemann sphere by $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the real line by $\mathbb{R}$ and the extended real line by $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. For the unit circle, its interior and its exterior we introduce the following notation:

$$\mathbb{T} = \{z: |z| = 1\}, \quad \mathbb{D} = \{z: |z| < 1\}, \quad \mathbb{E} = \{z: |z| > 1\}.$$

We will also use $I = [-1, 1]$, $\mathbb{R}^I = \hat{\mathbb{R}} \backslash I$ and $\mathbb{C}^I = \hat{\mathbb{C}} \backslash I$. Given positive bounded Borel measures μ on I and v on $[0, 2\pi]$, respectively, the inner product is defined as

$$\begin{aligned}\langle f, g \rangle &= \int_{-1}^{1} f(z)\overline{g(z)}\, d\mu(z), \quad \text{on } I,\\ &= \frac{1}{2\pi} \int_{0}^{2\pi} f(e^{\mathrm{i}\theta})\overline{g(e^{\mathrm{i}\theta})}\, dv(\theta), \quad \text{on } \mathbb{T}.\end{aligned}$$

In [1], the measure is assumed to be normalized, i.e. $\langle 1, 1 \rangle = 1$. The convergence results from Section 4 were proved under that assumption, but it is not difficult to see that they remain valid in the case of a nonnormalized measure. Therefore, in what follows we will drop the normalization assumption. The convergence results for the unit circle in Section 4 depend on the *divergence* to zero of the Blaschke products $B_n(z)$ defined in this section. The Blaschke factors as defined in [1] differ from the ones we define further on by a factor $-\bar{\beta}_n/|\beta_n|$. These unimodular constants are needed to ensure the convergence of the Blaschke products, but if they diverge to zero, they diverge with or without these constant factors. Since this type of divergent Blaschke products are considered in this paper, the presence of these constants is irrelevant for our results. We have used the definitions from [10].

Now we are ready to introduce the space of rational functions with real poles in $\mathbb{R}^I$. Let a sequence $\{\alpha_1, \alpha_2, \ldots\} \subset \mathbb{R}^I$ and a positive bounded Borel measure μ with $\operatorname{supp}(\mu) \subset I$ an infinite set be given (where $\operatorname{supp}(\mu)$ means the support of the measure, i.e. the smallest closed set whose complement has μ measure zero) and assume that $\mu' > 0$ a.e., where μ' is the Radon–Nikodym derivative of the measure μ with respect to the Lebesgue measure. Define factors

$$Z_n(z) = \frac{z}{1 - z/\alpha_n}, \quad n = 1, 2, \ldots$$

and basis functions

$$b_0 = 1, \quad b_n(z) = b_{n-1}(z) Z_n(z), \qquad n = 1, 2, \ldots \ .$$

Then we define the space of rational functions with poles in $\{\alpha_1, \ldots, \alpha_n\}$ as

$$\mathscr{L}_n = \operatorname{span}\{b_0, \ldots, b_n\}.$$

After orthonormalization of the basis $\{b_0, \ldots, b_n\}$ with respect to μ we obtain orthogonal rational functions $\{\varphi_0, \ldots, \varphi_n\}$, where we choose the leading coefficient k_n in the expansion $\varphi_n(z) = k_n b_n(z) + \cdots$ to be real. The ϕ_n will be uniquely determined once the sign of k_n is fixed. We will get back to this later on.

The orthogonal rational functions on the unit circle are defined similarly. Let a sequence of complex numbers $\{\beta_1, \beta_2, \ldots\} \subset \mathbb{D}$ and a positive bounded Borel measure v on $\mathbb{T}$ be given (again assume $\operatorname{supp}(v)$ is an infinite set and $v' > 0$ a.e.). Define the Blaschke factors

$$\zeta_n(z) = \frac{z - \beta_n}{1 - \bar{\beta}_n z}, \quad n = 1, 2, \ldots$$

and Blaschke products

$$B_0 = 1, \quad B_n(z) = \zeta_n(z) B_{n-1}(z), \qquad n = 1, 2, \ldots$$

and the space of rational functions on the unit circle,

$$\mathring{\mathscr{L}}_n = \operatorname{span}\{B_0, \ldots, B_n\}.$$

Note that the poles $\{1/\bar{\beta}_1, \ldots, 1/\bar{\beta}_n\}$ of a function $f \in \mathring{\mathscr{L}}_n$ are in $\mathbb{E}$. Orthonormalizing the basis $\{B_0, \ldots, B_n\}$ we obtain the orthogonal functions $\{\phi_0, \ldots, \phi_n\}$, where we choose the leading coefficient κ_n in $\phi_n(z) = \kappa_n B_n(z) + \cdots$ to be positive.

Now define the para-hermitian conjugate of a function $f(z)$ as

$$f_*(z) = \overline{f(1/\bar{z})}$$

and the superstar transform of ϕ_n as

$$\phi_n^*(z) = B_n(z)\phi_{n*}(z).$$

Note that ϕ_n^* is a function in $\overset{\circ}{\mathscr{L}}_n$.

3. Three-term recurrence

As in the polynomial case, orthogonal rational functions on (an interval of) the real line satisfy a three-term recurrence relation. We shall call the orthogonal rational function φ_n *singular* if $p_n(\alpha_{n-1}) = 0$, where p_n is the numerator polynomial of φ_n, and *regular* otherwise. If we put by convention $\alpha_{-1} = \alpha_0 = \infty$ then it can be shown [1] that the orthonormal functions φ_n satisfy the following three-term recurrence relation with the initial conditions $\varphi_{-1}(z) = 0$, $\varphi_0(z) = 1$ iff the system $\{\varphi_n\}$ is regular,

$$\varphi_n(z) = \left(E_n Z_n(z) + B_n \frac{Z_n(z)}{Z_{n-1}(z)}\right)\varphi_{n-1}(z) - \frac{E_n}{E_{n-1}} \frac{Z_n(z)}{Z_{n-2}(z)} \varphi_{n-2}(z). \tag{1}$$

If we take the coefficient E_n to be positive, then the functions ϕ_n will be uniquely determined. This amounts to fixing the sign of k_n. In the case of orthogonality on I and poles in $\mathbb{R}^I$ the regularity conditions are always satisfied. This follows from the following theorem, which can be found in [2].

Theorem 1. *Let φ_n be an orthogonal rational function on the interval $[-1, 1]$ with poles outside this interval. Then the zeros of φ_n are simple and contained in the open interval $(-1, 1)$.*

In [2], the measure was assumed to be absolutely continuous with respect to the Lebesgue measure, but the theorem still holds for arbitrary measures if the support of the measure is an infinite set (the proof proceeds along the same lines as the classical proof on the location of the zeros of orthogonal polynomials, see e.g. [9, p. 44]). Having established the regularity of the system $\{\varphi_n\}$ we know that recurrence relation (1) holds for all $n \geqslant 1$. It thus makes sense to study the asymptotic behaviour of the recurrence coefficients E_n and B_n as n tends to infinity.

4. Convergence results for the unit circle

In this section, we will recall some convergence results about orthogonal rational functions on the unit circle with poles in $\mathbb{E}$, as stated in [1]. In the following, locally uniform convergence in a region Ω will mean uniform convergence on compact subsets of Ω.

First we mention the following theorem from [1, Chapter 9].

Theorem 2. *Let* $\{\beta_1, \beta_2, \ldots\} \subset \mathbb{D}$ *be a sequence which is compactly included in* $\mathbb{D}$ *and* $\{\phi_0, \phi_1, \ldots\}$ *the orthonormal functions associated with the basis functions* $\{B_0, B_1, \ldots\}$ *as defined in Section* 2, *and let* v *be a positive bounded Borel measure on* $\mathbb{T}$ *satisfying the Erdős–Turán condition* $v' > 0$ *a.e. Then locally uniformly in* $\mathbb{D}$,

$$\lim_{n\to\infty} \frac{\phi_n(z)}{\phi_n^*(z)} = 0.$$

Regarding ratio asymptotics we have

$$\lim_{n\to\infty} \frac{\varepsilon_{n+1}\phi_{n+1}^*(z)(1-\bar{\beta}_{n+1}z)\sqrt{1-|\beta_n|^2}}{\varepsilon_n\phi_n^*(z)(1-\bar{\beta}_n z)\sqrt{1-|\beta_{n+1}|^2}} = 1,$$

where ε_n *is a unimodular constant such that* $\varepsilon_n\phi_n^*(0) > 0$, *i.e.* $\varepsilon_n = |\phi_n^*(0)|/\phi_n^*(0)$. *Again convergence is locally uniform in* $\mathbb{D}$.

If v satisfies the Szegő condition

$$\int_0^{2\pi} \log v'(\theta)\, d\theta > -\infty,$$

then we can define the Szegő function $\sigma(z)$ as

$$\sigma(z) = \exp\left\{\frac{1}{4\pi}\int_0^{2\pi} \frac{e^{\mathbf{i}\theta}+z}{e^{\mathbf{i}\theta}-z}\log v'(\theta)\, d\theta\right\}, \quad z \in \mathbb{D}.$$

In this case, we have the following strong convergence result from [1, Chapter 9].

Theorem 3. *Let* $\{\beta_1, \beta_2, \ldots\} \subset \mathbb{D}$ *be a sequence which is compactly included in* $\mathbb{D}$ *and* $\{\phi_0, \phi_1, \ldots\}$ *the orthonormal functions associated with the basis functions* $\{B_0, B_1, \ldots\}$ *as defined in Section* 2, *and let* v *be a positive bounded Borel measure on* $\mathbb{T}$ *satisfying Szegő's condition* $\log v' \in L^1[0, 2\pi]$. *Then locally uniformly in* $\mathbb{D}$,

$$\lim_{n\to\infty} \varepsilon_n \frac{\phi_n^*(z)(1-\bar{\beta}_n z)}{\sqrt{1-|\beta_n|^2}} = \frac{1}{\sigma(z)},$$

where ε_n *is as in Theorem* 2 *and* $\sigma(z)$ *is the Szegő function defined above.*

5. Relating the unit circle to the interval

In this section, we use x as the independent variable for the orthonormal rational functions $\varphi_n(x)$ on I and z for the functions $\phi_n(z)$ orthogonal on the unit circle. They can be related to each other in more or less the same way orthogonal polynomials on I can be related to orthogonal polynomials on the unit circle, see [9]. The relations in this section were derived in [10].

We denote the Joukowski transform $x = \frac{1}{2}(z + z^{-1})$ by $x = J(z)$, mapping the open unit disc $\mathbb{D}$ to the cut Riemann sphere $\mathbb{C}^I$ and the unit circle $\mathbb{T}$ to the interval I.

The inverse mapping is denoted by $z = J^{-1}(x)$ and is chosen so that $z \in \mathbb{D}$ if $x \in \mathbb{C}^I$. To the sequence $\{\alpha_1, \alpha_2, \ldots\} \subset \mathbb{R}^I$ we associate a sequence $\{\beta_1, \beta_2, \ldots\} \subset I$ such that $\beta_k = J^{-1}(\alpha_k)$, and a sequence $\{\tilde{\beta}_1, \tilde{\beta}_2, \ldots\}$ such that $\tilde{\beta}_{2k} = \tilde{\beta}_{2k-1} = \beta_k$. The corresponding Blaschke products and orthogonal functions are denoted by a tilde. Obviously, $\tilde{B}_{2k} = (B_k)^2$. The fact that $\{\alpha_n\} \subset \mathbb{R}^I$ implies that $\{\beta_n\}$ is actually in the *open* interval $(-1, 1)$ and thus in $\mathbb{D}$.

Next define the measure[1] v on $\mathbb{T}$ as

$$v(E) = \mu(\{\cos\theta, \theta \in E \cap [0, \pi)\}) + \mu(\{\cos\theta, \theta \in E \cap [\pi, 2\pi)\}). \tag{2}$$

This is sometimes written as $v(E) = \int_E |d\mu(\cos\theta)|$, but we prefer the less ambiguous notation. Using the Lebesgue decomposition of μ and the change-of-variables theorem (see e.g. [8, p. 153]) it is not difficult to see that

$$v'(\theta) = \mu'(\cos\theta)|\sin\theta|. \tag{3}$$

Then if $\{\tilde{\phi}_n\}$ is the orthonormal set associated with the measure v and the sequence $\{\tilde{\beta}_1, \tilde{\beta}_2, \ldots\}$, we have the following theorem.

Theorem 4. *Let $\{\varphi_n\}$ be a set of orthonormal rational functions on I and $\{\tilde{\phi}_n\}$ the corresponding set of functions orthogonal on $\mathbb{T}$ with poles and measure as defined above, then they are related by*

$$\varphi_n(x) = \delta_n(2\pi)^{-\frac{1}{2}}\left\{1 + \frac{\tilde{\phi}_{2n}(\beta_n)}{\tilde{\kappa}_{2n}}\right\}^{-\frac{1}{2}}\{B_{n*}(z)\tilde{\phi}_{2n}(z) + B_n(z)\tilde{\phi}_{2n*}(z)\},$$

where $x = J(z)$ and $\delta_n = \pm 1$ is such that the normalization of Section 2 holds.

Note that we have to double the multiplicity of every pole to obtain these results.

6. Ratio asymptotics

With the aid of Theorems 2 and 4, we are able to prove our main results about the convergence of the ratio of orthogonal rational functions on I. Of course more restrictive conditions on the location of the poles lead to more specific convergence results. In the sequel we will use the concept of asymptotic periodicity, which we define as follows: a sequence $\{\alpha_1, \alpha_2, \ldots\}$ is asymptotically periodic with period m if there exists a periodic sequence $\{\alpha_1^0, \alpha_2^0, \ldots\}$,

$$\alpha_{n+m}^0 = \alpha_n^0, \quad n = 1, 2, \ldots$$

[1] In [10], the measure μ was assumed to be absolutely continuous, but this can easily be extended to arbitrary positive Borel measures whose support is an infinite set. See also [3, p. 190] for the polynomial case.

such that

$$\lim_{n\to\infty} |\alpha_n - \alpha_n^0| = 0.$$

Now we shall state our first and most general result, where no other assumptions are made on the location of the poles than that they stay away from the boundary.

Theorem 5. *Assume the sequence* $A = \{\alpha_1, \alpha_2, \dots\} \subset \mathbb{R}^I$ *is bounded away from I and let* μ *be a positive bounded Borel measure with* $\operatorname{supp}(\mu) \subset I$ *an infinite set, which satisfies the Erdős–Turán condition* $\mu' > 0$ *a.e. If* $\{\varphi_n\}$ *are the orthonormal rational functions on I associated with A and* μ, *then locally uniformly in* $\mathbb{C}^I$ *we have*

$$\lim_{n\to\infty} \frac{z-\beta_{n+1}}{1-\beta_n z}\sqrt{\frac{1-\beta_n^2}{1-\beta_{n+1}^2}}\frac{\varphi_{n+1}(x)}{\varphi_n(x)} = 1,$$

where $z = J^{-1}(x)$ *and* $\beta_k = J^{-1}(\alpha_k)$ *for* $k = n, n+1$.

Proof. Define a measure v on $\mathbb{T}$ by (2) and then use Theorem 4 and the definitions from the previous sections to write

$$\frac{\varphi_{n+1}(x)}{\varphi_n(x)} = \frac{\delta_{n+1}}{\delta_n}\frac{1}{\zeta_{n+1}(z)}\frac{\tilde{\phi}^*_{2n+2}(z)}{\tilde{\phi}^*_{2n}(z)}\sqrt{\frac{1+\frac{\tilde{\phi}_{2n}(\beta_n)}{\tilde{\kappa}_{2n}}}{1+\frac{\tilde{\phi}_{2n+2}(\beta_{n+1})}{\tilde{\kappa}_{2n+2}}}\frac{\tilde{\phi}_{2n+2}(z)/\tilde{\phi}^*_{2n+2}(z)+1}{\tilde{\phi}_{2n}(z)/\tilde{\phi}^*_{2n}(z)+1}}.$$

Noting that $\tilde{\kappa}_{2k} = \tilde{\phi}^*_{2k}(\beta_k)$ for $k = n, n+1$ and using Theorem 2 we obtain

$$\lim_{n\to\infty} \frac{\tilde{\varepsilon}_{2n+2}}{\tilde{\varepsilon}_{2n}}\frac{\delta_n}{\delta_{n+1}}\frac{z-\beta_{n+1}}{1-\beta_n z}\sqrt{\frac{1-\beta_n^2}{1-\beta_{n+1}^2}}\frac{\varphi_{n+1}(x)}{\varphi_n(x)} = 1$$

locally uniformly in $\mathbb{C}^I$.

The asymptotic behaviour of the unimodular constant in front of this expression can be found as follows. From recurrence equation (1) we obtain the following expression:

$$E_{n+1} = \lim_{x\to\alpha_n} \frac{\varphi_{n+1}(x)}{\varphi_n(x)Z_{n+1}(x)}.$$

Then using the normalization $E_{n+1} > 0$ and the fact that convergence is uniform we find

$$\lim_{n\to\infty} \frac{\tilde{\varepsilon}_{2n+2}}{\tilde{\varepsilon}_{2n}}\frac{\delta_n}{\delta_{n+1}} = 1$$

(remember that all α_k and β_k are real, $|\alpha_k| > 1$, $|\beta_k| < 1$ and $\alpha_k\beta_k > 0$). This proves the theorem. □

If all poles are at infinity, then all $\beta_k = 0$ and we recover a well-known result about the asymptotic behaviour of the ratio of orthogonal polynomials on $[-1, 1]$, see e.g. [6].

As a corollary to our main theorem we consider the case of an asymptotically periodic pole sequence.

Corollary 1. *Assume the sequence* $A = \{\alpha_1, \alpha_2, \dots\} \subset \mathbb{R}^I$ *is asymptotically m-periodic with limiting values* $\{\alpha_i^0\}_{i=1}^m \subset \mathbb{R}^I$ *and assume the measure* μ *satisfies the conditions of Theorem* 5. *If* $\{\varphi_n\}$ *are the orthonormal rational functions on I associated with A and* μ, *then we have*

$$\lim_{n\to\infty} \frac{\varphi_{nm+i}(x)}{\varphi_{nm+i-1}(x)} = \frac{1-\beta_{i-1}^0 z}{z-\beta_i^0}\sqrt{\frac{1-(\beta_i^0)^2}{1-(\beta_{i-1}^0)^2}}, \quad i = 1, \dots, m$$

locally uniformly in $\mathbb{C}^I \backslash \{\alpha_i^0\}$, *where* $z = J^{-1}(x)$, $\beta_k^0 = J^{-1}(\alpha_k^0)$ *for* $k = i, i-1$ *and* $\alpha_0^0 = \alpha_m^0$.

If $m = 1$ we can easily obtain a more explicit expression for the limit function. We state the following result without proof, since this is a matter of simple algebra.

Corollary 2. *Assume the sequence* $A = \{\alpha_1, \alpha_2, \dots\} \subset \mathbb{R}^I$ *is such that* $\lim_{n\to\infty} \alpha_n = \alpha \in \mathbb{R}^I$ *and assume the measure* μ *satisfies the conditions of Theorem* 5. *If* $\{\varphi_n\}$ *are the orthonormal rational functions on I associated with A and* μ, *then we have for* $|\alpha| < \infty$

$$\lim_{n\to\infty} \frac{\varphi_{n+1}(x)}{\varphi_n(x)} = \frac{1-\alpha x}{x-\alpha} - \frac{\sqrt{(x^2-1)(\alpha^2-1)}}{x-\alpha}$$

locally uniformly in $\mathbb{C}^I \backslash \{\alpha\}$ *where the branch of the square root is chosen so that the expression on the right-hand side is greater than* 1 *in modulus for* $x \in \mathbb{C}^I$.

If $|\alpha| = \infty$ *then we have*

$$\lim_{n\to\infty} \frac{\varphi_{n+1}(x)}{\varphi_n(x)} = x + \sqrt{x^2-1}$$

locally uniformly in $\mathbb{C}^I$, *with the same convention for the branch of the square root.*

The relation between orthogonal rational functions on the unit circle and those on the interval can of course be used to find other asymptotic formulas as well. Using the strong convergence result of Theorem 3 we can prove the following theorem.

Theorem 6. *Assume the sequence* $A = \{\alpha_1, \alpha_2, \dots\} \subset \mathbb{R}^I$ *is bounded away from I and let* μ *be a positive bounded Borel measure with* $\text{supp}(\mu) \subset I$ *an infinite set, which satisfies the Szegő condition*

$$\int_{-1}^{1} \frac{\log \mu'(x)}{\sqrt{1-x^2}}\, dx > -\infty.$$

Let v *be defined by* (2) *and let* $\sigma(z)$ *be the associated Szegő function as defined in Section* 4. *If* $\{\varphi_n\}$ *are the orthonormal rational functions on I associated with A and* μ, *then locally uniformly in* $\mathbb{C}^I$ *we have*

$$\lim_{n\to\infty} c_n B_n(z) \frac{1-\beta_n z}{\sqrt{1-\beta_n^2}} \varphi_n(x) = \frac{1}{\sqrt{2\pi}\sigma(z)},$$

where $z = J^{-1}(x)$, $\beta_k = J^{-1}(\alpha_k)$ *and* $c_n = \pm 1$ *according to the normalization* $E_n > 0$.

In particular, we have

$$\lim_{n\to\infty} \varphi_n(x) = \infty$$

pointwise for $x \in \mathbb{C}^I$.

Proof. As in the proof of Theorem 5, $\varphi_n(x)$ can be written as

$$\varphi_n(x) = \delta_n \frac{\tilde{\phi}_{2n}^*(z)}{B_n(z)} (2\pi)^{-\frac{1}{2}} \left\{ 1 + \frac{\tilde{\phi}_{2n}(\beta_n)}{\tilde{\kappa}_{2n}} \right\}^{-\frac{1}{2}} \{\tilde{\phi}_{2n}(z)/\tilde{\phi}_{2n}^*(z) + 1\}.$$

Using Theorems 2 and 3 this yields

$$\lim_{n\to\infty} \frac{\tilde{\varepsilon}_{2n}}{\delta_n} B_n(z) \frac{1-\beta_n z}{\sqrt{1-\beta_n^2}} \varphi_n(x) = \frac{1}{\sqrt{2\pi}\sigma(z)}$$

locally uniformly in $\mathbb{C}^I$.

It follows from [10] that the function $\tilde{\phi}_{2n}(z)$ is real for real z. This implies also that $\tilde{\varepsilon}_{2n}$ is real, so we must have

$$\frac{\tilde{\varepsilon}_{2n}}{\delta_n} = \pm 1.$$

The last statement in the theorem follows from the fact that the Blaschke product $B_n(z)$ diverges to zero for $z \in \mathbb{D}$. □

Remark 1. If we take the normalization $k_n > 0$ instead of $E_n > 0$ for the functions $\varphi_n(x)$ then using $k_n = \lim_{x\to\alpha_n} \varphi_n(x)/b_n(x)$ and the fact that $\sigma(z)$ is real and positive for real z (because $v'(\theta)$ as defined by (3) is an even function), it is not difficult to show that c_n in the previous theorem tends to 1 with n so we can omit it from the statement.

7. Asymptotics for E_n and B_n

In this section, we wish to derive asymptotic formulas for the recurrence coefficients E_n and B_n. First we note that explicit formulas for the coefficients in terms of the orthogonal functions φ_n are given by

$$E_n = \lim_{x \to \alpha_{n-1}} \frac{\varphi_n(x)}{\varphi_{n-1}(x) Z_n(x)},$$

$$B_n = \lim_{x \to \alpha_{n-2}} \left(\frac{\varphi_n(x)}{\varphi_{n-1}(x)} \frac{Z_{n-1}(x)}{Z_n(x)} - E_n Z_{n-1}(x) \right).$$

These formulas are readily obtained from recurrence formula (1). Now we can use our main Theorem 5 to find the asymptotic formulas for E_n and B_n. The computations are cumbersome but straightforward and we omit the proof.

Theorem 7. *Under the assumptions of Theorem* 5, *the following relations hold in the sense that the ratio of the left-hand side and the right-hand side tends to* 1 *as n tends to infinity*,

$$E_n \sim 2 \frac{\sqrt{(1-\beta_{n-1}^2)(1-\beta_n^2)}(1-\beta_{n-1}\beta_n)}{(1+\beta_{n-1}^2)(1+\beta_n^2)},$$

$$B_n \sim -\sqrt{\frac{1-\beta_n^2}{1-\beta_{n-1}^2}} \frac{(1-\beta_{n-1}^2)(\beta_n+\beta_{n-2}) + 2\beta_{n-1}(1-\beta_n\beta_{n-2})}{(1+\beta_n^2)(1-\beta_{n-1}\beta_{n-2})},$$

where $\beta_k = J^{-1}(\alpha_k)$ *for* $k = n, n-1, n-2$.

It is interesting to note that for n large enough the coefficients E_n and B_n will only depend on, respectively, the last two or three poles. Another conclusion we can draw from this theorem is that E_n is bounded by $0 < E_n \leqslant 2$ for large enough n, while B_n can become arbitrarily large, depending on how close the poles come to the boundary of the interval. Take for example $\beta_n = \beta_{n-2} = 0$ and $\beta_{n-1} = \pm(1-\varepsilon)$ (where ε is a small positive number), then for large n we will have $B_n \approx \mp\sqrt{2/\varepsilon}$.

Of course we can write down explicit limits for the case of an asymptotically periodic pole sequence. In the special case $m = 1$ where all poles tend to a fixed pole $\alpha \in \mathbb{R}^I$ these expressions are simplified considerably and are given in the following corollary.

Corollary 3. *Under the assumptions of Corollary* 2 *we have the following convergence results for the recurrence coefficients*,

$$\lim_{n \to \infty} E_n = 2(1 - 1/\alpha^2),$$

$$\lim_{n \to \infty} B_n = -2/\alpha.$$

Again we note the correspondence with the polynomial case. If α equals infinity, the recurrence coefficients will behave asymptotically as the recurrence coefficients in the well-known recurrence formula for orthogonal polynomials on I, see e.g. [9, p. 310] for the case of an absolutely continuous measure satisfying Szegő's condition, and [6, p. 212] for the general situation.

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